Final exam (online) — Functional Analysis (WIFA–08)

Monday 6 April 2020, 18.45h–21.45h CEST (plus 30 minutes for uploading)

University of Groningen

Instructions

1. Only references to the lecture notes and slides are allowed. References to other sources are *not* allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
4. Write both your name and student number on the answer sheets!
5. This exam comes in two versions. Both versions consist of five problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
6. Upload your work as a single PDF file in your personal Nestor dropbox folder.

Version 1 (for odd student numbers)

Problem 1 (5 + 10 + 10 = 25 points)

A function $f : [0,1] \to \mathbb{K}$ is called *Lipschitz-continuous* if there exists a constant $c \geq 0$, which may depend on f, such that

$$|f(x) - f(y)| \leq c|x - y| \quad \text{for all } x, y \in [0,1].$$

Denote the set of all such functions by $\mathcal{L}([0,1],\mathbb{K})$. Clearly, $\mathcal{L}([0,1],\mathbb{K}) \subset \mathcal{C}([0,1],\mathbb{K})$.

(a) Show that $\mathcal{L}([0,1],\mathbb{K})$ is a linear subspace of $\mathcal{C}([0,1],\mathbb{K})$.

(b) For $f \in \mathcal{L}([0,1],\mathbb{K})$ we define

$$\|f\|_{\mathcal{L}} = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} \;:\; x, y \in [0,1],\, x \neq y\right\}.$$

Prove that $\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{L}([0,1],\mathbb{K})$.

(c) Is the norm $\|\cdot\|_{\mathcal{L}}$ equivalent to the sup-norm $\|\cdot\|_\infty$?

Problem 2 (5 + 10 + 10 + 5 = 30 points)

Equip the space $\mathcal{C}([-1,1],\mathbb{K})$ with the norm $\|f\|_\infty = \sup_{x\in[-1,1]} |f(x)|$, and consider the following linear operator:

$$T : \mathcal{C}([-1,1],\mathbb{K}) \to \mathcal{C}([-1,1],\mathbb{K}), \quad Tf(x) = f(|x|).$$

(a) Compute the operator norm of T.

(b) Show that $\lambda = 0$ and $\lambda = 1$ are eigenvalues of T.

(c) Prove that if $\lambda \notin \{0,1\}$, then $\lambda \in \rho(T)$.

(d) Is T compact?

Problem 3 (10 points)

It is given that for all $f \in \mathcal{C}([0,\pi],\mathbb{K})$ the following boundary value problem has a unique solution $u \in \mathcal{C}^2([0,\pi],\mathbb{K})$:

$$u''(x) + u(x) = f(x), \quad 0 < x < \pi, \quad u(0) = u'(\pi) = 0.$$

Prove that there exists a constant $C \geq 0$ such that

$$\|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \leq C\|f\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the usual supremum-norm on $\mathcal{C}([0,\pi],\mathbb{K})$.

Hint: you may use without proof that the space

$$X = \left\{u \in \mathcal{C}^2([0,\pi],\mathbb{K}) \;:\; u(0) = u'(\pi) = 0\right\}$$

equipped with the norm $\|u\|_2 = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$ is a Banach space.

Problem 4 (10 + 5 = 15 points)

(a) Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Prove that the following statements are equivalent:

(i) T is bounded;

(ii) if (x_n) is a sequence in X such that $x_n \to 0$ and $Tx_n \to y$, then $y = 0$.

(b) Now assume that X is a Hilbert space over $\mathbb{C}$ and that the linear operator $T : X \to X$ satisfies the following property:

$$(Tx, z) = (x, Tz) \quad \text{for all} \quad x, z \in X,$$

where $(\cdot\,,\,\cdot)$ denotes the innerproduct on X. Use part (a) to prove that T is bounded.

Problem 5 (10 points)

On the linear space $\mathbb{R}^2$ we take the following norm:

$$\|x\|_1 = |x_1| + |x_2|, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Consider the following linear maps:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = 7x_1 - 3x_2 \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}, \quad g(x) = 7x_1 + 5x_2.$$

Show that:

(i) $\|f\| = \|g\|$, but $f \neq g$;

(ii) there exists a nontrivial subspace $V \subset \mathbb{R}^2$ such that $f(x) = g(x)$ for all $x \in V$.

Discuss the implication for the Hahn-Banach Theorem.

End of test ("version 1", 90 points)

Version 2 (for even student numbers)

Problem 1 (5 + 10 + 10 = 25 points)

A function $f : [0,1] \to \mathbb{K}$ is called *Lipschitz-continuous* if there exists a constant $c \geq 0$, which may depend on f, such that

$$|f(x) - f(y)| \leq c|x - y| \quad \text{for all } x, y \in [0,1].$$

Denote the set of all such functions by $\mathcal{L}([0,1],\mathbb{K})$. Clearly, $\mathcal{L}([0,1],\mathbb{K}) \subset \mathcal{C}([0,1],\mathbb{K})$.

(a) Show that $\mathcal{L}([0,1],\mathbb{K})$ is a linear subspace of $\mathcal{C}([0,1],\mathbb{K})$.

(b) For $f \in \mathcal{L}([0,1],\mathbb{K})$ we define

$$\|f\|_{\mathcal{L}} = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} \; : \; x, y \in [0,1],\, x \neq y\right\}.$$

Prove that $\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{L}([0,1],\mathbb{K})$.

(c) Is the norm $\|\cdot\|_{\mathcal{L}}$ equivalent to the sup-norm $\|\cdot\|_\infty$?

Problem 2 (5 + 10 + 10 + 5 = 30 points)

Equip the space $\mathcal{C}([-1,1],\mathbb{K})$ with the norm $\|f\|_\infty = \sup_{x\in[-1,1]} |f(x)|$, and consider the following linear operator:

$$T : \mathcal{C}([-1,1],\mathbb{K}) \to \mathcal{C}([-1,1],\mathbb{K}), \quad Tf(x) = f(x) + f(-x).$$

(a) Compute the operator norm of T.

(b) Show that $\lambda = 0$ and $\lambda = 2$ are eigenvalues of T.

(c) Prove that if $\lambda \notin \{0, 2\}$, then $\lambda \in \rho(T)$.

(d) Is T compact?

Problem 3 (10 points)

It is given that for all $f \in \mathcal{C}([0,1],\mathbb{K})$ the following initial value problem has a unique solution $u \in \mathcal{C}^1([0,1],\mathbb{K})$:

$$u'(x) + 2x \cdot u(x) = f(x), \quad 0 < x < 1, \quad u(0) = 0.$$

Prove that there exists a constant $C \geq 0$ such that

$$\|u\|_\infty + \|u'\|_\infty \leq C\|f\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the usual supremum-norm on $\mathcal{C}([0,1],\mathbb{K})$.

Hint: you may use without proof that the space

$$X = \{u \in \mathcal{C}^1([0,1],\mathbb{K}) \; : \; u(0) = 0\}$$

equipped with the norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$ is a Banach space.

Problem 4 (10 + 5 = 15 points)

(a) Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Prove that the following statements are equivalent:

  (i) T is bounded;

  (ii) if (x_n) is a sequence in X such that $x_n \to 0$ and $Tx_n \to y$, then $y = 0$.

(b) Now assume that X is a Hilbert space over $\mathbb{C}$ and that the linear operator $T : X \to X$ satisfies the following property:

$$|(Tx, z)| \leq \|x\| \, \|z\| \quad \text{for all} \quad x, z \in X,$$

where $(\cdot\,, \cdot)$ denotes the innerproduct on X. Use part (a) to prove that T is bounded.

Problem 5 (10 points)

On the linear space $\mathbb{R}^2$ we take the following norm:

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Consider the following linear maps:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = 5x_1 + 3x_2 \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}, \quad g(x) = 3x_1 + 5x_2.$$

Show that:

(i) $\|f\| = \|g\|$, but $f \neq g$;

(ii) there exists a nontrivial subspace $V \subset \mathbb{R}^2$ such that $f(x) = g(x)$ for all $x \in V$.

Discuss the implication for the Hahn-Banach Theorem.

End of test ("version 2", 90 points)

Solution of problem 1, version 1 and 2 (5 + 10 + 10 = 25 points)

(a) Let $f, g \in \mathcal{L}([0,1],\mathbb{K})$ and $\lambda \in \mathbb{K}$. There exist constants $c, d \geq 0$ such that

$$\begin{aligned}|(f+g)(x)-(f+g)(y)| &= |f(x)+g(x)-f(y)-g(y)| \\ &\leq |f(x)-f(y)|+|g(x)-g(y)| \\ &\leq c|x-y|+d|x-y| \\ &= (c+d)|x-y|,\end{aligned}$$

which shows that $f+g \in \mathcal{L}([0,1],\mathbb{K})$.
(3 points)

In addition,

$$|(\lambda f)(x)-(\lambda f)(y)| = |\lambda|\,|f(x)-f(y)| \leq c|\lambda|\,|x-y|,$$

which shows that $\lambda f \in \mathcal{L}([0,1],\mathbb{K})$.
(2 points)

(b) It is clear that $\|f\|_{\mathcal{L}} \geq 0$ for all $f \in \mathcal{L}([0,1],\mathbb{K})$. Conversely, if $\|f\|_{\mathcal{L}} = 0$, then $f(0) = 0$ and

$$\frac{|f(x)-f(y)|}{|x-y|} = 0 \quad \text{for all } x, y \in [0,1] \text{ and } x \neq y,$$

so that $f(x) = f(y)$ for all $x, y \in [0,1]$. In particular, $f(x) = f(0) = 0$ for all $x \in [0,1]$.
(2 points)

If $f \in \mathcal{L}([0,1],\mathbb{K})$ and $\lambda \in \mathbb{K}$, then

$$\begin{aligned}\|\lambda f\|_{\mathcal{L}} &= |\lambda f(0)| + \sup\left\{\frac{|\lambda f(x)-\lambda f(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &= |\lambda|\,|f(0)| + \sup\left\{|\lambda|\cdot\frac{|f(x)-f(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &= |\lambda|\,|f(0)| + |\lambda|\sup\left\{\frac{|f(x)-f(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &= |\lambda|\,\|f\|_{\mathcal{L}}.\end{aligned}$$

(4 points)

Finally, if $f, g \in \mathcal{L}([0,1],\mathbb{K})$ and $\lambda \in \mathbb{K}$, then

$$\begin{aligned}\|f+g\|_{\mathcal{L}} &= |f(0)+g(0)| + \sup\left\{\frac{|f(x)+g(x)-f(y)-g(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &\leq |f(0)|+|g(0)| + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|}+\frac{|g(x)-g(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &\leq |f(0)|+|g(0)| + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &\qquad + \sup\left\{\frac{|g(x)-g(y)|}{|x-y|} \;:\; x,y\in[0,1], x\neq y\right\} \\ &= \|f\|_{\mathcal{L}} + \|g\|_{\mathcal{L}}.\end{aligned}$$

(4 points)

(b) Define a sequence of functions $f_n : [0,1] \to \mathbb{K}$ by

$$f_n(x) = \begin{cases} nx & \text{if } x \in [0, \frac{1}{n}], \\ 1 & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

Note that

$$\frac{|f_n(x) - f_n(y)|}{|x-y|} = \begin{cases} n & \text{if } x, y \in [0, \frac{1}{n}] \text{ and } x \neq y, \\ \frac{|nx-1|}{|x-y|} = n \cdot \frac{|x - \frac{1}{n}|}{|x-y|} < n & \text{if } x \in [0, \frac{1}{n}], y \in (\frac{1}{n}, 1], \\ 0 & \text{if } x, y \in [\frac{1}{n}, 1] \text{ and } x \neq y. \end{cases}$$

Therefore, $\|f_n\|_{\mathcal{L}} = n$ for all $n \in \mathbb{N}$.
(7 points)

On the other hand, $\|f_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Hence, there is no constant $C > 0$ such that $\|f\|_{\mathcal{L}} \leq C\|f\|_\infty$ for all $f \in \mathcal{L}([0,1], \mathbb{K})$, which implies that the norms $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|_\infty$ are not equivalent.
(3 points)

Solution of problem 2, version 1 (5 + 10 + 10 + 5 = 30 points)

(a) For any $f \in \mathcal{C}([-1,1],\mathbb{K})$ we have that

$$\|Tf\|_\infty = \sup_{x\in[-1,1]} |f(|x|)| = \sup_{x\in[0,1]} |f(x)| \leq \sup_{x\in[-1,1]} |f(x)| = \|f\|_\infty.$$

(3 points)

Now take $f \in \mathcal{C}([-1,1],\mathbb{K})$ such that $f(x) = 1$ for all $x \in [-1,1]$. Then $\|f\|_\infty = 1$ and $\|Tf\|_\infty = 1$. We conclude that the operator norm of T is given by

$$\|T\| = \sup_{f\in\mathcal{C}([-1,1],\mathbb{K}),f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 1.$$

(2 points)

(b) We have that $Tf = 0$ if and only if

$$f(|x|) = 0 \quad \text{for all} \quad x \in [-1,1],$$

or, equivalently,

$$f(x) = 0 \quad \text{for all} \quad x \in [0,1].$$

(2 points)

Clearly, there exist nonzero functions f that satisfy this condition. For example, take the function

$$f(x) = \begin{cases} x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

We conclude that $\lambda = 0$ is an eigenvalue of T.
(3 points)

We have that $Tf = f$ if and only if

$$f(|x|) = f(x) \quad \text{for all} \quad x \in [-1,1].$$

(2 points)

Clearly, there exist nonzero functions f that satisfy this condition. For example, take the function $f(x) = x^2$. We conclude that $\lambda = 1$ is an eigenvalue of T.
(3 points)

(c) Assume that $\lambda \notin \{0,1\}$. If $(T-\lambda)f = g$, then

$$f(|x|) - \lambda f(x) = g(x) \quad \text{for all} \quad x \in [-1,1],$$

or, equivalently,

$$\begin{cases} f(x) - \lambda f(x) = g(x) & \text{if } 0 \leq x \leq 1, \\ f(-x) - \lambda f(x) = g(x) & \text{if } -1 \leq x < 0. \end{cases}$$

Solving for f gives

$$\begin{aligned} f(x) &= (T-\lambda)^{-1} g(x) \\ &:= \begin{cases} \dfrac{1}{1-\lambda} g(x) & \text{if } 0 \leq x \leq 1, \\ \dfrac{1}{\lambda}\big(-g(x) + f(-x)\big) = -\dfrac{1}{\lambda} g(x) + \dfrac{1}{\lambda(1-\lambda)} g(-x) & \text{if } -1 \leq x < 0. \end{cases} \end{aligned}$$

(5 points)

For all $x \in [0,1]$ it follows that

$$|(T-\lambda)^{-1}g(x)| \leq \begin{cases} \dfrac{1}{|1-\lambda|}\|g\|_\infty & \text{if } 0 \leq x \leq 1, \\ \dfrac{1}{|\lambda|}\|g\|_\infty + \dfrac{1}{|\lambda(1-\lambda)|}\|g\|_\infty & \text{if } -1 \leq x < 0. \end{cases}$$

so that

$$\|(T-\lambda)^{-1}g(x)\|_\infty \leq C\|g\|_\infty,$$

where

$$C = \max\left\{\frac{1}{|1-\lambda|}, \frac{1}{|\lambda|} + \frac{1}{|\lambda(1-\lambda)|}\right\}.$$

Therefore, $(T-\lambda)^{-1}$ is bounded and hence $\lambda \in \rho(T)$.
(5 points)

Alternative argument. Since $T-\lambda$ is bijective and $\mathcal{C}([-1,1],\mathbb{K})$ is a Banach space, it follows by a corollary of the Open Mapping Theorem that $(T-\lambda)^{-1}$ is bounded. Therefore, $\lambda \in \rho(T)$.
(5 points)

(d) Recall the following result: if T is compact, then for any eigenvalue $\lambda \neq 0$ the corresponding eigenspace $\ker\,(T-\lambda)$ is finite-dimensional.

For $\lambda = 1$ we have that

$$\operatorname{span}\{x^{2n} \,:\, n \in \mathbb{N}\} \subset \ker\,(T-\lambda),$$

which shows that $\ker\,(T-\lambda)$ is *not* finite-dimensional. We conclude that T is not compact.
(5 points; only 4 points when argument is given for $\lambda = 0$)

Note. It is important that the argument is given for $\lambda = 1$. It is easy to give an example of a compact operator T for which $\ker T$ is infinite-dimensional.

Solution of problem 2, version 2 (5 + 10 + 10 + 5 = 30 points)

(a) For any $f \in \mathcal{C}([-1,1],\mathbb{K})$ we have that

$$\|Tf\|_\infty = \sup_{x\in[-1,1]} |f(x)+f(-x)| \leq \sup_{x\in[-1,1]} |f(x)| + \sup_{x\in[-1,1]} |f(-x)| = 2\|f\|_\infty.$$

(3 points)

Now take $f \in \mathcal{C}([-1,1],\mathbb{K})$ such that $f(x) = 1$ for all $x \in [-1,1]$. Then $\|f\|_\infty = 1$ and $\|Tf\|_\infty = 2$. We conclude that the operator norm of T is given by

$$\|T\| = \sup_{f\in\mathcal{C}([-1,1],\mathbb{K}),f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 2.$$

(2 points)

(b) We have that $Tf = 0$ if and only if

$$f(x) + f(-x) = 0 \quad \text{for all} \quad x \in [-1,1].$$

(2 points)

Clearly, there exist nonzero functions f that satisfy this condition. For example, take the function $f(x) = x$. We conclude that $\lambda = 0$ is an eigenvalue of T.
(3 points)

We have that $Tf = 2f$ if and only if

$$f(x) + f(-x) = 2f(x) \quad \text{for all} \quad x \in [-1,1],$$

or, equivalently,

$$f(x) = f(-x) \quad \text{for all} \quad x \in [-1,1].$$

(2 points)

Clearly, there exist nonzero functions f that satisfy this condition. For example, take the function $f(x) = x^2$. We conclude that $\lambda = 2$ is an eigenvalue of T.
(3 points)

(c) Assume that $\lambda \notin \{0,2\}$. If $(T-\lambda)f = g$, then

$$(1-\lambda)f(x) + f(-x) = g(x) \quad \text{for all} \quad x \in [-1,1].$$

Replacing x by $-x$ gives the equation

$$f(x) + (1-\lambda)f(-x) = g(-x) \quad \text{for all} \quad x \in [-1,1].$$

Therefore, for any $x \in [-1,1]$ can find $f(x)$ by solving the following system:

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} f(x) \\ f(-x) \end{pmatrix} = \begin{pmatrix} g(x) \\ g(-x) \end{pmatrix}.$$

There is a unique solution if and only if $\lambda \notin \{0,2\}$. In that case, we find that

$$f(x) = \frac{1-\lambda}{\lambda(\lambda-2)} g(x) - \frac{1}{\lambda(\lambda-2)} g(-x).$$

(5 points)

For all $x \in [-1, 1]$ it follows that

$$|(T - \lambda)^{-1} g(x)| = \left| \frac{1 - \lambda}{\lambda(\lambda - 2)} \right| |g(x)| + \left| \frac{1}{\lambda(\lambda - 2)} \right| |g(-x)| \leq C \|g\|_\infty,$$

where

$$C = \left| \frac{1 - \lambda}{\lambda(\lambda - 2)} \right| + \left| \frac{1}{\lambda(\lambda - 2)} \right|.$$

Therefore,

$$\|(T - \lambda)^{-1} g(x)\|_\infty \leq C \|g\|_\infty,$$

which means that $(T - \lambda)^{-1}$ is bounded and hence $\lambda \in \rho(T)$.
(5 points)

Alternative argument. Since $T - \lambda$ is bijective and $\mathcal{C}([-1, 1], \mathbb{K})$ is a Banach space, it follows by a corollary of the Open Mapping Theorem that $(T - \lambda)^{-1}$ is bounded. Therefore, $\lambda \in \rho(T)$.
(5 points)

(d) Recall the following result: if T is compact, then for any eigenvalue $\lambda \neq 0$ the corresponding eigenspace ker $(T - \lambda)$ is finite-dimensional.

For $\lambda = 2$ we have that

$$\text{span}\,\{x^{2n} \,:\, n \in \mathbb{N}\} \subset \ker\,(T - \lambda),$$

which shows that ker $(T - \lambda)$ is *not* finite-dimensional. We conclude that T is not compact.
(5 points; only 4 points when argument is given for $\lambda = 0$)

Note. It is important that the argument is given for $\lambda = 1$. It is easy to give an example of a compact operator T for which ker T is infinite-dimensional.

Solution of problem 3, version 1 (10 points)

Define the following linear operator

$$T : X \to \mathcal{C}([0,\pi],\mathbb{K}), \quad Tu = u'' + u.$$

Note that T is bounded:

$$\begin{aligned} \|Tu\|_\infty &= \sup_{x\in[0,\pi]} |u''(x) + u(x)| \\ &\leq \|u''\|_\infty + \|u\|_\infty \\ &\leq \|u\|_2. \end{aligned}$$

(3 points)

Since the spaces $(X, \|\cdot\|_2)$ and $(\mathcal{C}([0,\pi],\mathbb{K}), \|\cdot\|_\infty)$ are Banach spaces and it is given that T is bijective, it follows by a corollary of the Open Mapping Theorem that the operator $T^{-1} : \mathcal{C}([0,\pi],\mathbb{K}) \to X$ is bounded. This means that there exists a constant $C \geq 0$ such that

$$\|T^{-1}f\|_2 \leq C\|f\|_\infty$$

for all $f \in \mathcal{C}([0,\pi],\mathbb{K})$.
(5 points)

Finally, note that u is a solution of the given boundary value problem if and only if $Tu = f$, or, equivalently, $u = T^{-1}f$. The boundedness of T^{-1} now gives the desired inequality:

$$\|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty = \|u\|_2 = \|T^{-1}f\|_2 \leq C\|f\|_\infty.$$

(2 points)

Solution of problem 3, version 2 (10 points)

Define the following linear operator

$$T : X \to \mathcal{C}([0,1],\mathbb{K}), \quad Tu = u' + 2x\,u.$$

Note that T is bounded:

$$\begin{aligned} \|Tu\|_\infty &= \sup_{x\in[0,1]} |u'(x) + 2x\,u(x)| \\ &\leq \|u'\|_\infty + 2\|u\|_\infty \\ &\leq 2\|u\|_1. \end{aligned}$$

(3 points)

Since the spaces $(X, \|\cdot\|_1)$ and $(\mathcal{C}([0,1],\mathbb{K}), \|\cdot\|_\infty)$ are Banach spaces and it is given that T is bijective, it follows by a corollary of the Open Mapping Theorem that the operator $T^{-1} : \mathcal{C}([0,1],\mathbb{K}) \to X$ is bounded. This means that there exists a constant $C \geq 0$ such that

$$\|T^{-1}f\|_1 \leq C\|f\|_\infty$$

for all $f \in \mathcal{C}([0,1],\mathbb{K})$.
(5 points)

Finally, note that u is a solution of the given initial value problem if and only if $Tu = f$, or, equivalently, $u = T^{-1}f$. The boundedness of T^{-1} now gives the desired inequality:

$$\|u\|_\infty + \|u'\|_\infty = \|u\|_1 = \|T^{-1}f\|_1 \leq C\|f\|_\infty.$$

(2 points)

Solution of problem 4, version 1 (10 + 5 = 15 points)

(a) *Proof of (i) ⇒ (ii).* Assume that T is bounded. Let (x_n) be a sequence such that $x_n \to 0$ and $Tx_n \to y$. Then it follows that

$$\|y\| = \|y - Tx_n + Tx_n\| \leq \|y - Tx_n\| + \|Tx_n\| \leq \|y - Tx_n\| + \|T\| \, \|x_n\|.$$

Since the right-hand side tends to zero, it follows that $y = 0$.
(5 points)

Proof of (ii) ⇒ (i). Assume that $x_n \to x$ and $Tx_n \to y$. Introduce the new sequence $z_n = x_n - x$. Then it follows that $z_n \to 0$ and $Tz_n \to y - Tx$. By assumption it follows that $y - Tx = 0$ so that $y = Tx$. We conclude that the graph of T is closed. Since X and Y are Banach spaces we can apply the Closed Graph Theorem with $V = X$ to conclude that T is bounded.
(5 points)

(b) Let $z \in X$ be arbitrary, and let (x_n) be a sequence in X such that $x_n \to 0$ and $Tx_n \to y$. On the one hand, we have that

$$(Tx_n, z) = (x_n, Tz) \to 0.$$

On the other hand, we have that

$$(Tx_n, z) \to (y, z).$$

(3 points)

By uniqueness of limits, we conclude that $(y, z) = 0$. Since $z \in X$ was arbitrary, it follows that $y \in X^\perp = \{0\}$ so that $y = 0$. By part (a) we conclude that T is bounded.
(2 points)

Solution of problem 4, version 2 (10 + 5 = 15 points)

(a) Identical to version 1.

(b) Let $z \in X$ be arbitrary, and let (x_n) be a sequence in X such that $x_n \to 0$ and $Tx_n \to y$. On the one hand, we have that

$$|(Tx_n, z)| \leq \|x_n\| \, \|z\| \to 0.$$

On the other hand, we have that

$$(Tx_n, z) \to (y, z).$$

(3 points)

By uniqueness of limits, we conclude that $(y, z) = 0$. Since $z \in X$ was arbitrary, it follows that $y \in X^\perp = \{0\}$ so that $y = 0$. By part (a) we conclude that T is bounded.
(2 points)

Solution of problem 5, version 1 (10 points)

For all $x = (x_1, x_2) \in \mathbb{R}^2$ we have that

$$|f(x)| = |7x_1 - 3x_2| \leq 7|x_1| + 3|x_2| \leq 7\|x\|_1,$$
$$|g(x)| = |7x_1 + 5x_2| \leq 7|x_1| + 5|x_2| \leq 7\|x\|_1.$$

For $x = (1, 0)$ we have $\|x\|_1 = 1$ and $|f(x)| = |g(x)| = 7$. We conclude that

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_1} = 7 \quad \text{and} \quad \|g\| = \sup_{x \neq 0} \frac{|g(x)|}{\|x\|_1} = 7.$$

(4 points)

For $x = (0, 1)$ we have that $f(x) = -3$ and $g(x) = 5$, so $f \neq g$.
(1 point)

With $V = \operatorname{span}\{(1, 0)\}$ we have $f(x) = g(x)$ for all $x \in V$.
(1 point)

Define the linear map $h : V \to \mathbb{R}$ by $h(x) = f(x)$. It easily follows that $\|h\| = 7$. Both f and g are norm preserving extensions of h. This implies that norm preserving extensions, of which the *existence* is guaranteed by the Hahn-Banach Theorem, need not be unique.
(4 points)

Solution of problem 5, version 2 (10 points)

For all $x = (x_1, x_2) \in \mathbb{R}^2$ we have that

$$|f(x)| = |5x_1 + 3x_2| \leq 5|x_1| + 3|x_2| \leq 8\|x\|_\infty,$$
$$|g(x)| = |3x_1 + 5x_2| \leq 3|x_1| + 5|x_2| \leq 8\|x\|_\infty.$$

For $x = (1, 1)$ we have $\|x\|_\infty = 1$ and $|f(x)| = |g(x)| = 8$. We conclude that

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_\infty} = 8 \quad \text{and} \quad \|g\| = \sup_{x \neq 0} \frac{|g(x)|}{\|x\|_\infty} = 8.$$

(4 points)

For $x = (1, 0)$ we have that $f(x) = 3$ and $g(x) = 5$, so $f \neq g$.
(1 point)

With $V = \operatorname{span}\{(1, 1)\}$ we have $f(x) = g(x)$ for all $x \in V$.
(1 point)

Define the linear map $h : V \to \mathbb{R}$ by $h(x) = f(x)$. It easily follows that $\|h\| = 8$. Both f and g are norm preserving extensions of h. This implies that norm preserving extensions, of which the *existence* is guaranteed by the Hahn-Banach Theorem, need not be unique.
(4 points)